where the (i', j', k') denote the interpolation points, shown in Figure 8.29b. The degrees of freedom provided by (8.115) are $(p+1)(p+2)q/2$.

As pointed out by Graglia et al. [51], these basis functions provide element-to-element tangential continuity even in the curvilinear case. The explicit expressions for the pth/first-order triangular prism element are available in the literature [71].

8.6.2.4 Numerical Examples To demonstrate the performance of higher-order three-dimensional vector elements, we consider two examples here. The first example is the electromagnetic wave scattering by an open cavity whose aperture size is $1.0\lambda \times 1.0\lambda$ and whose depth is 4.0λ [71]. The aperture coincides with an infinitely large ground plane. This problem can be analyzed by using the finite element method to formulate the field inside the cavity and the boundary integral method to represent the field outside the cavity and then coupling the interior and exterior fields by the field continuity conditions. This method is discussed in detail in Chapter 10. Because the formulation of this method is exact, any error in the solution has to be introduced by numerical discretization. To examine the accuracy, we consider the radar cross section (RCS) and define the root-mean-square (RMS) error in RCS as

$$\text{RMS} = \sqrt{\frac{1}{N_s} \sum_{i=1}^{N_s} |\sigma_{\text{ref}} - \sigma_{\text{cal}}|^2}, \tag{8.116}$$

where σ_{cal} denotes the calculated RCS and σ_{ref} denotes the reference solution, both measured in dB, and N_s is the number of sampling points, which are the angles of incidence here. The reference solution in this case is obtained using the fourth-order tetrahedral elements with an overly dense mesh such that the solution does not change anymore when either the order of elements or the mesh density is increased. Figure 8.30 displays the RMS error in the monostatic RCS of the cavity as a function of the number of unknowns, calculated using the first-, second-, third-, and fourth-order vector tetrahedral elements. It is evident that for the same number of unknowns, higher-order elements produce more accurate results. For a desired accuracy, the number of unknowns required for the higher-order elements is much smaller than that for lower-order elements, as expected.

The second example is a circular cavity having the same height and radius. The cavity is discretized into triangular prism elements. The percentage errors in the computed resonant frequencies versus the number of unknowns are given in Figure 8.31 for the first-, second-, and third-order triangular prism elements (the same order is used here for both the transverse and longitudinal fields). The error is averaged over the first eight resonant frequencies. In general, higher-order elements give better results and converge faster. Asymptotically, for the pth-order elements the error decreases in $O(h^{2p})$, as in the two-dimensional case. However, the accuracy of the results also depends heavily on the quality of the finite element mesh and the modeling of curved surfaces in this case.

8.7 HIGHER-ORDER HIERARCHICAL VECTOR ELEMENTS

As mentioned in the preceding section, in addition to interpolatory vector elements, there is another type of higher-order vector element, which is called hierarchical vector element. This type of element has also been used widely in the finite element analysis of electromagnetic fields because of its two distinct advantages. First, it permits the use of basis functions of different orders in a single finite element mesh, which can facilitate the implementation of adaptive p-refinement [72–74]. Second, it provides a foundation in the development of multigrid and multilevel solvers or preconditioners for an efficient solution of finite element equations [75–78]. These two advantages are obtained because in a higher-order hierarchical element the basis functions include explicitly those of a lower-order element.

In this section we describe hierarchical vector basis functions for triangular and tetrahedral elements based on the work by Webb [56]. A systematic formulation is also presented by Zhu and Cangellaris [78].

8.7.1 Scalar Hierarchical Basis Functions

Because this is the first time we touch the topic of hierarchical basis functions, we start with scalar ones because they are relatively simple and easy to understand, although they are not as widely used as interpolatory basis functions discussed in Chapters 3–5. The hierarchical basis functions constructed for finite element analysis have to satisfy the following three requirements:

1. They should be linearly independent and complete to the desired order. Therefore, the number of hierarchical basis functions for a given order is the same as the number of interpolatory basis functions.

2. They should make it easy to enforce the interelement continuity for the expanded quantity in the finite element formulation. The interelement continuity refers to the continuity at the nodes, across the edges, and across the faces of the elements.

3. The basis functions of a given order should explicitly include those of lower orders. In other words, the hierarchical basis functions of order p should retain all the basis functions of order $p-1$ and then add a few new ones necessary to make them complete to order p.

The first two requirements are the same as those for interpolatory basis functions. It is the third that distinguishes hierarchical from interpolatory ones. In fact, these requirements are less restrictive than those for interpolatory basis functions. Consequently, there are many more choices in formulating hierarchical basis functions, as illustrated below.

8.7.1.1 Line Element Consider a line element, whose two end nodes are labeled as 1 and 2 with the associated simplex coordinates denoted as ξ_1 and ξ_2, which are related by $\xi_1 + \xi_2 = 1$. The first-order hierarchical basis functions are simply

$$N_i = \xi_i \qquad i = 1, 2, \tag{8.117}$$

which are the same as the first-order interpolatory basis functions. The second-order hierarchical basis functions consist of those in (8.117) and a new one

$$N_3 = \xi_1 \xi_2, \tag{8.118}$$

which contains the necessary second-order term. Note that this function vanishes at nodes 1 and 2, hence its inclusion does not affect the interelement continuity at these nodes. To construct the third-order hierarchical basis functions, we would retain those in (8.117) and (8.118) and add a new function that contains the necessary third-order term. There are several choices for this new function, and the obvious ones are $\xi_1^2\xi_2$ and $\xi_1\xi_2^2$, both of which vanish at nodes 1 and 2. We can also choose their linear combinations as long as the combinations do not reduce the order of the function. One of such combinations is

$$N_4 = \xi_1\xi_2(\xi_1 - \xi_2), \tag{8.119}$$

which yields an odd function along the element. Again, this function vanishes at nodes 1 and 2 so that it does not impact the interelement continuity. Following this approach, we can systematically construct hierarchical basis functions of any order, just by adding a new function that contains any of the terms $\xi_1^q\xi_2^{p-q}$ $(q = 1, 2, \ldots, p-1)$ or their linear combinations, such as

$$N_{p+1} = \xi_1\xi_2(\xi_1 - \xi_2)^{p-2}. \tag{8.120}$$

Clearly, a pth-order hierarchical line element has $p+1$ basis functions.

8.7.1.2 Triangular Element Let ξ_1, ξ_2, and ξ_3 denote the simplex coordinates of a triangular element (Fig. 4.3), which satisfy the relation $\xi_1 + \xi_2 + \xi_3 = 1$. The first-order hierarchical basis functions are simply

$$N_i^{\mathrm{n}} = \xi_i \qquad i = 1, 2, 3, \tag{8.121}$$

which are the same as the first-order interpolatory basis functions. The superscript n denotes that these basis functions are associated with the nodes. The second-order hierarchical basis functions consist of those in (8.121) and three new functions:

$$N_{ij}^{\mathrm{e},2} = \xi_i\xi_j \qquad (i,j) = (1,2), (2,3), (3,1), \tag{8.122}$$

where the first superscript e denotes that these basis functions are associated with the edges, the second superscript 2 denotes the order of the basis function, and the subscript ij denotes the two nodes of the edge associated with the basis function. The superscripts and subscripts are used to make the meaning of $N_{ij}^{\mathrm{e},2}$ obvious. Note that $N_{ij}^{\mathrm{e},2}$ vanishes at all the nodes and all the edges except for the edge (i,j). Therefore, the interelement continuity can easily be enforced. The third-order hierarchical basis functions consist of those in (8.121) and (8.122) and another three new functions associated with the edges:

$$N_{ij}^{\mathrm{e},3} = \xi_i\xi_j(\xi_i - \xi_j) \qquad (i,j) = (1,2), (2,3), (3,1). \tag{8.123}$$

Similar to the case of a line element, there are many other choices that are equally useful as long as they contain the necessary third-order terms and they vanish at all the nodes and all the edges except for the associated edge (i,j) so that their use does not complicate the enforcement of interelement continuity. So far, we have nine basis functions in (8.121), (8.122), and (8.123); however, a third-order triangular element should have ten linearly independent basis functions. The tenth basis function is given by

$$N_{123}^{\mathrm{f},3} = \xi_1\xi_2\xi_3, \tag{8.124}$$

where the superscript f denotes that this basis function is associated with the face. This function vanishes at all the nodes and all the edges. Following this approach, we can systematically construct hierarchical basis functions of any order for a triangular element. Similar to an interpolatory element, a pth-order hierarchical triangular element has $(p+1)(p+2)/2$ linearly independent basis functions, of which three are associated with the nodes, $3(p-1)$ are associated with the edges, and $(p-1)(p-2)/2$ are associated with the face.

8.7.1.3 Tetrahedral Element The approach just described can be used easily to construct hierarchical basis functions for a tetrahedral element (Fig. 5.1). The first-order hierarchical basis functions are

$$N_i^{\rm n} = \xi_i \qquad i = 1, 2, 3, 4, \tag{8.125}$$

which are the same as the first-order interpolatory basis functions. The second-order hierarchical basis functions consist of those in (8.125) and six new functions:

$$N_{ij}^{{\rm e},2} = \xi_i \xi_j \qquad (i,j) = (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \tag{8.126}$$

for the six edges. The third-order hierarchical basis functions consist of those in (8.125) and (8.126), another six edge-associated functions:

$$N_{ij}^{{\rm e},3} = \xi_i \xi_j (\xi_i - \xi_j) \qquad (i,j) = (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \tag{8.127}$$

and four face-associated functions:

$$N_{ijk}^{{\rm f},3} = \xi_i \xi_j \xi_k \qquad (i,j,k) = (1,2,3), (1,3,4), (1,4,2), (2,4,3), \tag{8.128}$$

one for each of the four faces. Note that $N_{ijk}^{{\rm f},3}$ vanishes at all the nodes, all the edges, and all the faces except for the face defined by nodes (i, j, k). So far, all are the same as for a triangular element. However, a tetrahedral element is different from a triangular element when its order is greater than 3. For example, a fourth-order tetrahedral element would contain, in addition to those associated with the nodes, edges, and faces, a basis function

$$N_{1234}^{{\rm v},4} = \xi_1 \xi_2 \xi_3 \xi_4 \tag{8.129}$$

associated with the volume of the element, which vanishes at all the nodes, all the edges, and all the faces. Similar to an interpolatory element, a pth-order hierarchical tetrahedral element has $(p+1)(p+2)(p+3)/6$ linearly independent basis functions, of which four are associated with the nodes, $6(p-1)$ are associated with the edges, $2(p-1)(p-2)$ are associated with the faces, and $(p-1)(p-2)(p-3)/6$ are associated with the volume.

The numerical implementation of the finite element method using scalar hierarchical basis functions is nearly identical to that with interpolatory basis functions because the arrangements of the basis functions are similar in both cases. If we want to use hierarchical basis functions of different orders in a single mesh, all we have to do is to enforce the interelement continuity between two neighboring elements that have a different order. This can be done by simply removing higher-order basis functions on the edges and faces in the higher-order element that are shared by the lower-order elements.

8.7.2 Separation of Gradient and Rotational Basis Functions

Before we extend the approach to formulating hierarchical basis functions from the scalar to the vector case, let us first discuss more clearly the issue of mixed order versus full order in a vector element. As mentioned earlier, a distinct feature of the Nédélec vector basis functions is that these functions are of a mixed order. The variation of these functions in the direction of the vector is one order less than the variation in the direction normal to the vector. This mixed-order formulation makes the basis functions more suitable to represent the divergence-free electric and magnetic fields, and enhances the efficiency of the finite element solution by reducing the number of basis functions.

To see this more clearly, consider a vector triangular element in Figure 8.2 or a tetrahedral element in Figure 8.12. It is rather obvious to see that fully linear vector basis functions associated with the edge that connects nodes i and j consist of $\xi_i \nabla \xi_i$ and $\xi_i \nabla \xi_i$, which are linearly independent and satisfy all the requirements for vector basis functions. These two functions have nontrivial divergence and curl. However, we can construct other pairs of linearly independent basis functions from their linear combinations. In particular, consider the following two combinations:

$$\mathbf{W}_{ij} = \xi_i \nabla \xi_i - \xi_i \nabla \xi_i \tag{8.130}$$

$$\mathbf{V}_{ij} = \xi_i \nabla \xi_i + \xi_i \nabla \xi_i, \tag{8.131}$$

which are linearly independent. The first combination yields a rotational or a solenoidal function because $\nabla \cdot \mathbf{W}_{ij} = 0$. The second combination can be written as a pure gradient function

$$\mathbf{V}_{ij} = \nabla(\xi_i \xi_i), \tag{8.132}$$

which is irrotational because $\nabla \times \mathbf{V}_{ij} = \nabla \times \nabla(\xi_i \xi_i) = 0$. Both $\mathbf{W}_{ij}$ and $\mathbf{V}_{ij}$ still form a complete first-order basis that can be used to represent a vector field, say the electric field $\mathbf{E}$. However, because $\mathbf{V}_{ij}$ does not contribute to the representation of $\nabla \times \mathbf{E}$, it can be dropped so that the representations of $\mathbf{E}$ and $\nabla \times \mathbf{E}$ become more balanced and the number of basis functions is reduced. Consequently, only $\mathbf{W}_{ij}$ remains, which forms a mixed first-order (or an incomplete first-order) because $\mathbf{W}_{ij}$ is constant in the direction tangential to its direction and linear in the normal direction.

This discussion also suggests a general approach to separating gradient basis functions from a complete set of basis functions. The first step is to take the gradient on the scalar basis functions of order $p+1$ to obtain the gradient basis functions of order p. The second step is to remove the gradient basis functions from the full vector basis functions of order p. The remaining parts are either rotational functions or rotational-like functions because they are dominated by rotational functions.

8.7.3 Vector Hierarchical Basis Functions

We are now ready to develop hierarchical vector basis functions for any finite elements. These basis functions should satisfy the following three requirements:

1. They should be linearly independent and complete to the desired order. In a full-order case, the polynomials should be complete to the desired order in all directions, whereas in a mixed-order case, the order of the polynomials in the tangential direction is reduced by 1.

2. They should make it easy to enforce the interelement tangential continuity for the expanded vector in the finite element formulation. The interelement tangential continuity refers to the tangential continuity across the edges and the faces of the elements.

3. The basis functions of a given order should explicitly include those of lower orders and can be constructed simply by adding a few new basis functions necessary to make them complete to the desired order.

Similar to the case for scalar hierarchical basis functions, there is an unlimited number of choices in formulating vector hierarchical basis functions. In fact, many vector hierarchical basis functions have been proposed in the past—presented below is just one of the many.

8.7.3.1 Triangular Element Before we construct specific gradient and rotational vector basis functions, let us first identify the complete set of vector basis functions for a given order, say p. For a triangular element (Fig. 8.2) characterized by simplex coordinates (ξ_1, ξ_2, ξ_3), there are two types of basis functions. One is associated with the edges, which consists of

$$P_1(\xi_i, \xi_j)\xi_i \nabla \xi_j + P_2(\xi_i, \xi_j)\xi_j \nabla \xi_i \tag{8.133}$$

for edge (i, j), where P_1 and P_2 are polynomials of degree $p - 1$. Altogether, there are $3(p+1)$ such linearly independent functions. These functions have a tangential component only along edge (i, j). The other type, which occurs when $p > 1$, is associated with the face, which is the interior of the triangular element. These basis functions have the form

$$P_1(\xi_1, \xi_2, \xi_3)\xi_2\xi_3 \nabla \xi_1 + P_2(\xi_1, \xi_2, \xi_3)\xi_3\xi_1 \nabla \xi_2 + P_3(\xi_1, \xi_2, \xi_3)\xi_1\xi_2 \nabla \xi_3, \tag{8.134}$$

where P_1, P_2, and P_3 are polynomials of degree $p - 2$. Among the three, only two are linearly independent; hence altogether there are $(p - 1)(p + 1)$ such linearly independent functions. These functions do not have a tangential component along any edges.

Now to construct the first-order vector basis functions, we take the gradient of the second-order scalar basis functions in (8.122) to first find the gradient basis functions as

$$\mathbf{N}_{ij}^{\mathrm{e,g,1}} = \nabla(\xi_i \xi_j) \qquad (i, j) = (1, 2), (2, 3), (3, 1), \tag{8.135}$$

where the superscript g stands for gradient. We then remove them from the complete first-order vector basis to find the rotational basis functions as

$$\mathbf{N}_{ij}^{\mathrm{e,r,1}} = \xi_i \nabla \xi_j - \xi_j \nabla \xi_i \qquad (i, j) = (1, 2), (2, 3), (3, 1), \tag{8.136}$$

where the superscript r stands for rotational or rotational-like. The functions in (8.136) are the mixed first-order basis functions and when combined with those in (8.135) form the full first-order basis functions for a triangular element.

To construct the second-order vector basis functions, we take the gradient of the third-order scalar basis functions in (8.123) and (8.124) to first find the gradient basis functions as

$$\mathbf{N}_{ij}^{\mathrm{e,g,2}} = \nabla[\xi_i \xi_j (\xi_i - \xi_j)] \qquad (i, j) = (1, 2), (2, 3), (3, 1) \tag{8.137}$$

$$\mathbf{N}_{ijk}^{\mathrm{f,g,2}} = \nabla(\xi_i \xi_j \xi_k) \qquad (i, j, k) = (1, 2, 3). \tag{8.138}$$

We then remove them from the complete second-order basis to find the rotational-like basis functions as

$$\mathbf{N}_{ijk}^{\mathrm{f,r,2}} = \xi_j(\xi_k \nabla \xi_i - \xi_i \nabla \xi_k) + \xi_i(\xi_k \nabla \xi_j - \xi_j \nabla \xi_k) \qquad (i, j, k) = (1, 2, 3), (2, 3, 1). \tag{8.139}$$

Note that we have discarded the third combination corresponding to $(i, j, k) = (3, 1, 2)$ because it is not linearly independent. The basis functions in (8.135)–(8.139) form the full second-order basis, which is reduced to the mixed second-order when the gradient basis functions in (8.137) and (8.138) are removed.

Similarly, to construct the third-order vector basis functions, we take the gradient of the fourth-order scalar basis functions to first find the gradient basis functions as

$$\mathbf{N}_{ij}^{\mathrm{e,g},3} = \nabla[\xi_i \xi_j (\xi_i - \xi_j)^2] \qquad (i, j) = (1, 2), (2, 3), (3, 1) \tag{8.140}$$

$$\mathbf{N}_{ijk}^{\mathrm{f,g},3} = \nabla[\xi_i \xi_j \xi_k (\xi_i - \xi_j)] \qquad (i, j, k) = (1, 2, 3), (2, 3, 1). \tag{8.141}$$

Removing these from the complete third-order basis yields the rotational-like basis functions as

$$\mathbf{N}_{ijk}^{\mathrm{f,r},3} = \xi_i \xi_j (\xi_j \nabla \xi_k - \xi_k \nabla \xi_j) \qquad (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2). \tag{8.142}$$

The basis functions in (8.135)–(8.142) form the full third-order basis, which is reduced to the mixed third-order when the gradient basis functions in (8.140) and (8.141) are removed.

This approach can be extended to construct hierarchical vector basis functions for a triangular element of any full- or mixed-order. For a full pth-order element, there are $(p+1)(p+2)$ basis functions, and for the corresponding mixed-order element, the number of basis functions is reduced to $p(p+2)$.

8.7.3.2 Tetrahedral Element A complete set of vector basis functions for a tetrahedral element (Fig. 8.12) characterized by simplex coordinates $(\xi_1, \xi_2, \xi_3, \xi_4)$ consists of three types of functions. The first type is associated with the edges and consists of

$$P_1(\xi_i, \xi_j)\xi_i \nabla \xi_j + P_2(\xi_i, \xi_j)\xi_j \nabla \xi_i \tag{8.143}$$

for edge (i, j), where P_1 and P_2 are polynomials of degree $p - 1$. Altogether, there are $6(p+1)$ such linearly independent functions. These functions have a tangential component only along edge (i, j) and on the two faces joined by the edge. The second type, which occurs when $p > 1$, is associated with the face and has the form

$$P_1(\xi_i, \xi_j, \xi_k)\xi_j \xi_k \nabla \xi_i + P_2(\xi_i, \xi_j, \xi_k)\xi_k \xi_i \nabla \xi_j + P_3(\xi_i, \xi_j, \xi_k)\xi_i \xi_j \nabla \xi_k \tag{8.144}$$

for face (i, j, k), where P_1, P_2, and P_3 are polynomials of degree $p - 2$. Among the three, only two are linearly independent on each face; hence altogether there are $4(p-1)(p+1)$ such linearly independent functions. These functions do not have a tangential component along any edges and on any faces except for face (i, j, k). The third type, which occurs when $p > 2$, is associated with the volume of the tetrahedral element and has the form

$$\begin{aligned} &P_1(\xi_1, \xi_2, \xi_3, \xi_4)\xi_2 \xi_3 \xi_4 \nabla \xi_1 + P_2(\xi_1, \xi_2, \xi_3, \xi_4)\xi_3 \xi_4 \xi_1 \nabla \xi_2 \\ &\quad + P_3(\xi_1, \xi_2, \xi_3, \xi_4)\xi_4 \xi_1 \xi_2 \nabla \xi_3 + P_4(\xi_1, \xi_2, \xi_3, \xi_4)\xi_1 \xi_2 \xi_3 \nabla \xi_4, \end{aligned} \tag{8.145}$$

where P_1, P_2, P_3, and P_4 are polynomials of degree $p - 3$. Among the four, only three are linearly independent; hence altogether there are $(p-2)(p-1)(p+1)/2$ such linearly independent functions. These functions do not have a tangential component along any edges and on any faces.

To construct the first-order vector basis functions, we start with the second-order scalar basis functions in (8.126) and take the gradient to find the gradient basis functions as

$$\mathbf{N}_{ij}^{\mathrm{e,g,1}} = \nabla(\xi_i\xi_j) \qquad (i,j) = (1,2),(1,3),(1,4),(2,3),(2,4),(3,4). \quad (8.146)$$

We then remove them from the complete first-order vector basis to find the rotational basis functions as

$$\mathbf{N}_{ij}^{\mathrm{e,r,1}} = \xi_i\nabla\xi_j - \xi_j\nabla\xi_i \qquad (i,j) = (1,2),(1,3),(1,4),(2,3),(2,4),(3,4), \quad (8.147)$$

which are the mixed first-order vector basis functions. The full first-order basis functions consist of those in both (8.146) and (8.147).

To construct the second-order vector basis functions, we take the gradient of the third-order scalar basis functions in (8.127) and (8.128) to first find the gradient basis functions as

$$\mathbf{N}_{ij}^{\mathrm{e,g,2}} = \nabla[\xi_i\xi_j(\xi_i - \xi_j)] \qquad (i,j) = (1,2),(1,3),(1,4),(2,3),(2,4),(3,4) \quad (8.148)$$

$$\mathbf{N}_{ijk}^{\mathrm{f,g,2}} = \nabla(\xi_i\xi_j\xi_k) \qquad (i,j,k) = (1,2,3),(1,3,4),(1,4,2),(2,4,3). \quad (8.149)$$

We then remove them from the complete second-order basis to find the rotational-like basis functions as

$$\mathbf{N}_{ijk}^{\mathrm{f,r,2}} = \xi_j(\xi_k\nabla\xi_i - \xi_i\nabla\xi_k) + \xi_i(\xi_k\nabla\xi_j - \xi_j\nabla\xi_k)$$
$$(i,j,k) = \begin{Bmatrix} (1,2,3),(2,3,1) \\ (1,3,4),(3,4,1) \\ (1,4,2),(4,2,1) \\ (2,4,3),(4,3,2) \end{Bmatrix}, \quad (8.150)$$

with two on each of the four faces. The basis functions in (8.146)–(8.150) form the full second-order basis, which is reduced to the mixed second-order when the gradient basis functions in (8.148) and (8.149) are removed.

Similarly, to construct the third-order vector basis functions, we take the gradient of the fourth-order scalar basis functions to first find the gradient basis functions as

$$\mathbf{N}_{ij}^{\mathrm{e,g,3}} = \nabla[\xi_i\xi_j(\xi_i - \xi_j)^2] \qquad (i,j) = (1,2),(1,3),(1,4),(2,3),(2,4),(3,4) \quad (8.151)$$

$$\mathbf{N}_{ijk}^{\mathrm{f,g,3}} = \nabla[\xi_i\xi_j\xi_k(\xi_i - \xi_j)] \qquad (i,j,k) = \begin{Bmatrix} (1,2,3),(2,3,1) \\ (1,3,4),(3,4,1) \\ (1,4,2),(4,2,1) \\ (2,4,3),(4,3,2) \end{Bmatrix} \quad (8.152)$$

$$\mathbf{N}_{ijk\ell}^{\mathrm{v,g,3}} = \nabla(\xi_i\xi_j\xi_k\xi_\ell) \qquad (i,j,k,\ell) = (1,2,3,4). \quad (8.153)$$

Removing these from the complete third-order basis yields the rotational-like basis functions as

$$\mathbf{N}_{ijk}^{\mathrm{f,r,3}} = \xi_i\xi_j(\xi_j\nabla\xi_k - \xi_k\nabla\xi_j) \qquad (i,j,k) = \begin{Bmatrix} (1,2,3),(2,3,1),(3,1,2) \\ (1,3,4),(3,4,1),(4,1,3) \\ (1,4,2),(4,2,1),(2,1,4) \\ (2,4,3),(4,3,2),(3,2,4) \end{Bmatrix} \quad (8.154)$$

$$\mathbf{N}_{ijk\ell}^{\mathrm{v,r,3}} = \xi_i\xi_j\xi_k\nabla\xi_\ell \qquad (i,j,k,\ell) = (1,2,3,4),(2,3,4,1),(3,4,1,2). \quad (8.155)$$

The basis functions in (8.146)–(8.155) form the full third-order vector basis, which is reduced to the mixed third-order when the gradient basis functions in (8.151)–(8.153) are removed.

This approach can be extended to construct hierarchical vector basis functions for a tetrahedral element of any full- or mixed-order. For a full pth-order element, there are $(p+1)(p+2)(p+3)/2$ basis functions, and for the corresponding mixed-order element, the number of basis functions is reduced to $p(p+2)(p+3)/2$.

Once vector hierarchical basis functions are constructed, their numerical implementation in the finite element method is rather straightforward, although it is more complicated than the scalar case. If mixed-order hierarchical basis functions are employed, the implementation can follow directly that with mixed-order interpolatory basis functions because the number of basis functions and their arrangements (association with the edges, faces, and volumes) can be made identical. Similar to the case for interpolatory vector elements, an important issue in the numerical implementation of hierarchical vector elements is to make sure that the vector basis functions associated with the edges and the faces are consistent among those sharing the same edge and the same face. If we want to use hierarchical basis functions of different orders in a single mesh, all we have to do is to enforce the interelement tangential continuity between two neighboring elements that have a different order. This can be done by simply removing higher-order basis functions on the edges and faces in the higher-order element that are shared by the lower-order elements. This simple treatment makes hierarchical basis functions ideally suited for adaptive p-refinement.

The hierarchical vector basis functions can also be constructed in a similar manner for quadrilateral, hexahedral, and triangular prism elements. This is left for the reader as an exercise.

Exercise 8.5 Construct hierarchical gradient and rotational-like vector basis functions for a quadrilateral element.

Exercise 8.6 Construct hierarchical gradient and rotational-like vector basis functions for a triangular prism element.

8.7.4 Orthogonality of Hierarchical Vector Basis Functions

Although hierarchical vector elements enjoy two distinct advantages over their interpolatory counterpart, these advantages come with a price. As the order increases, hierarchical vector basis functions, especially once discretized, look more and more similar to each other, although they are still linearly independent. Consequently, the finite element matrices obtained using these basis functions become more and more ill-conditioned. When the finite element equations are solved iteratively, the number of iterations increases significantly and the efficiency of the finite element solution is compromised. The remedy is to perform an orthogonalization of higher-order basis functions so that the elemental matrices produced by these functions become well conditioned and when assembled they produce a better conditioned system matrix. The two functions, say $\mathbf{A}$ and $\mathbf{B}$, are said to be orthogonal on an element Ω^e when they satisfy the condition

$$\mathbf{A} \perp \mathbf{B} = \int_{\Omega^e} \mathbf{A} \cdot \mathbf{B} \, d\Omega = 0. \tag{8.156}$$

Although this orthogonalization can be performed on each element, doing so would result in different basis functions for different elements, which would make it difficult, if not impossible, to enforce the interelement continuity. Therefore, this orthogonalization is usually performed on an equilateral element. The resulting basis functions are orthogonal on this element, but only approximately so for elements that are not equilateral. Nevertheless, when most elements are close to equilateral ones, the orthogonalization can still significantly improve the condition of the resulting finite element matrix.

There are many ways to perform orthogonalization on higher-order hierarchical basis functions. By using the Gram-Schmidt orthogonalization method, Webb developed a set of orthogonal hierarchical basis functions for an equilateral tetrahedral element complete to order 3 [56]. For convenience, we present all the basis functions here even though some of the basis functions have to be repeated. The index combinations are omitted because they are the same as those in Section 8.7.3. Instead, we denote the number of basis functions generated from each expression in the parentheses behind the expression. The orthogonal basis functions for an equilateral triangular element can be extracted easily from the results given because a triangle is simply one of the four faces of a tetrahedron. For each order, only additional basis functions are given, and it is understood that the complete set includes these additional functions and all the prior functions.

Mixed first-order:

$$\mathbf{N}_{ij}^{\mathrm{e,r,1}} = \xi_i \nabla \xi_j - \xi_j \nabla \xi_i \quad (6)$$

Full first-order:

$$\mathbf{N}_{ij}^{\mathrm{e,g,1}} = \nabla(\xi_i \xi_j) \quad (6)$$

Mixed second-order:

$$\mathbf{N}_{ijk}^{\mathrm{f,r,2}} = \xi_j(\xi_k \nabla \xi_i - \xi_i \nabla \xi_k) + \xi_i(\xi_k \nabla \xi_j - \xi_j \nabla \xi_k) \quad (8)$$

Full second-order:

$$\mathbf{N}_{ij}^{\mathrm{e,g,2}} = \nabla[\xi_i \xi_j(\xi_i - \xi_j)] \quad (6)$$

$$\mathbf{N}_{ijk}^{\mathrm{f,g,2}} = \nabla(\xi_i \xi_j \xi_k) \quad (4)$$

Mixed third-order:

$$\begin{aligned}\mathbf{N}_{ijk}^{\mathrm{f,r,3}} &= 699\nabla[\xi_i \xi_j \xi_k(\xi_j - \xi_i)] + 2719(\xi_i - \xi_j)\xi_i \xi_j \nabla \xi_k \\ &\quad + 86\xi_k(\xi_i \nabla \xi_j - \xi_j \nabla \xi_i) \quad (12)\end{aligned}$$

$$\mathbf{N}_{ijk\ell}^{\mathrm{v,r,3}} = \xi_j \xi_k(\xi_\ell + 3\xi_i)\nabla \xi_i + \xi_k \xi_i(\xi_\ell + 3\xi_j)\nabla \xi_j + \xi_i \xi_j(\xi_\ell + 3\xi_k)\nabla \xi_k \quad (1)$$

$$\mathbf{N}_{ijk\ell}^{\mathrm{v,r,3}} = \xi_k \xi_\ell(\xi_j \nabla \xi_i - \xi_i \nabla \xi_j) + \xi_j \xi_\ell(\xi_k \nabla \xi_i - \xi_i \nabla \xi_k) \quad (1)$$

$$\mathbf{N}_{ijk\ell}^{\mathrm{v,r,3}} = \xi_\ell \xi_i(\xi_k \nabla \xi_j - \xi_j \nabla \xi_k) \quad (1)$$

Full third-order:

$$\mathbf{N}_{ij}^{\mathrm{e,g,3}} = \nabla[\xi_i \xi_j(15\xi_i^2 + 15\xi_j^2 - 30\xi_i \xi_j - 2)] \quad (6)$$

$$\mathbf{N}_{ijk}^{\mathrm{f,g,3}} = \nabla[\xi_i \xi_j \xi_k(\xi_i - \xi_j)] \quad (8)$$

$$\mathbf{N}_{ijk\ell}^{\mathrm{v,g,3}} = \nabla(\xi_i \xi_j \xi_k \xi_\ell) \quad (1)$$

Another orthogonalization was proposed by Sun et al. [57] also for an equilateral tetrahedron by using the reduction of the number of negative eigenvalues and the condition number of the system matrix as criteria. This results in a set of somewhat different orthogonal basis functions:

Mixed first-order:

$$\mathbf{N}_{ij}^{\mathrm{e,r,1}} = \xi_i \nabla\xi_j - \xi_j \nabla\xi_i \quad (6)$$

Full first-order:

$$\mathbf{N}_{ij}^{\mathrm{e,g,1}} = \nabla(\xi_i\xi_j) \quad (6)$$

Mixed second-order:

$$\mathbf{N}_{ijk}^{\mathrm{f,r,2}} = \xi_j(\xi_k \nabla\xi_i - \xi_i \nabla\xi_k) + \xi_i(\xi_k \nabla\xi_j - \xi_j \nabla\xi_k) \quad (4)$$

$$\mathbf{N}_{ijk}^{\mathrm{f,r,2}} = \xi_k(\xi_j \nabla\xi_i - \xi_i \nabla\xi_j) \quad (4)$$

Full second-order:

$$\mathbf{N}_{ij}^{\mathrm{e,g,2}} = \nabla[\xi_i\xi_j(\xi_i - \xi_j)] \quad (6)$$

$$\mathbf{N}_{ijk}^{\mathrm{f,g,2}} = \nabla(\xi_i\xi_j\xi_k) \quad (4)$$

Mixed third-order:

$$\mathbf{N}_{ijk}^{\mathrm{f,r,3}} = (\xi_j - \xi_k)\xi_j\xi_k \nabla\xi_i + (\xi_k - \xi_i)\xi_k\xi_i \nabla\xi_j + (\xi_i - \xi_j)\xi_i\xi_j \nabla\xi_k \quad (4)$$

$$\mathbf{N}_{ijk}^{\mathrm{f,r,3}} = (80\xi_j - 212\xi_k + 393\xi_i)\xi_j\xi_k \nabla\xi_i + (212\xi_k - 80\xi_i + 393\xi_j)\xi_k\xi_i \nabla\xi_j$$
$$-292(\xi_i - \xi_j)\xi_i\xi_j \nabla\xi_k \quad (4)$$

$$\mathbf{N}_{ijk}^{\mathrm{f,r,3}} = (186\xi_j - 124\xi_k - 131\xi_i)\xi_j\xi_k \nabla\xi_i + (124\xi_k - 168\xi_i + 393\xi_j)\xi_k\xi_i \nabla\xi_j$$
$$-(44\xi_i + 44\xi_j - 262\xi_k)\xi_i\xi_j \nabla\xi_k \quad (4)$$

$$\mathbf{N}_{ijk\ell}^{\mathrm{v,r,3}} = \xi_i\xi_j(\xi_k \nabla\xi_\ell - \xi_\ell \nabla\xi_k) + \xi_k\xi_\ell(\xi_j \nabla\xi_i - \xi_i \nabla\xi_j) \quad (1)$$

$$\mathbf{N}_{ijk\ell}^{\mathrm{v,r,3}} = \xi_\ell\xi_i(\xi_k \nabla\xi_j - \xi_j \nabla\xi_k) \quad (1)$$

$$\mathbf{N}_{ijk\ell}^{\mathrm{v,r,3}} = \xi_j\xi_k(\xi_1 \nabla\xi_\ell - \xi_\ell \nabla\xi_i) \quad (1)$$

Full third-order:

$$\mathbf{N}_{ij}^{\mathrm{e,g,3}} = \nabla[\xi_i\xi_j(362\xi_i^2 + 362\xi_j^2 - 504\xi_i\xi_j - 117\xi_i - 117\xi_j - 16)] \quad (6)$$

$$\mathbf{N}_{ijk}^{\mathrm{f,g,3}} = \nabla[\xi_i\xi_j\xi_k(\xi_i + \xi_j - 2\xi_k)] \quad (4)$$

$$\mathbf{N}_{ijk}^{\mathrm{f,g,3}} = \nabla[\xi_i\xi_j\xi_k(\xi_i - \xi_j)] \quad (4)$$

$$\mathbf{N}_{ijk\ell}^{\mathrm{v,g,3}} = \nabla(\xi_i\xi_j\xi_k\xi_\ell) \quad (1)$$

The development of orthogonal higher-order hierarchical vector basis functions continues to draw some interest even today because there is an unlimited number of choices in constructing and orthogonalizing such functions [58–70].