COMPUTER METHODS IN APPLIED MECHANICS AND ENGINEERING 55 (1986) 339–348
NORTH-HOLLAND

MODERATE-DEGREE TETRAHEDRAL QUADRATURE FORMULAS

Patrick KEAST
Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada B3H 4H8

Received 17 June 1985
Revised manuscript received 19 August 1985

Quadrature formulas of degrees 4 to 8 for numerical integration over the tetrahedron are constructed. The formulas are fully symmetric with respect to the tetrahedron, and in some cases are the minimum point rules with this symmetry.

1. Introduction

In [4] Jinyun has described formulas for numerical integration over the three-dimensional simplex, or tetrahedron, with polynomial degrees 2 to 6. The formulas possess full tetrahedral symmetry, thus making them attractive for use in finite element codes, where it is clearly preferable that the orientation of the tetrahedral elements should not affect the distribution of the quadrature evaluation points. Although previous formulas also possessed these symmetry properties, they had other disadvantages such as the use of excessive numbers of function evaluations (e.g., [7]) or availability only for low degree (e.g., [3, 9]).

The rules in [4] were computed by first postulating a structure for the quadrature rule, and then attempting to solve the nonlinear moment equations for the evaluation points and weights. The purpose of this note is to demonstrate a systematic method for deriving valid rule structures and for computing the corresponding formulas.

The technique used is described more fully in [6]. Another systematic method appeared in [2]. The formulas obtained by Jinyun will be given as special cases here, and some formulas will be obtained using fewer points. We will list valid structures for rules of degrees 1 to 10, and will compute formulas of degrees 4 to 8.

2. Derivation of formulas

Following the notation of [5, 8], let $T(\alpha_1, \alpha_2, \alpha_3; \alpha_4)$, $\alpha_i \geq 0$, $\sum_{i=1}^{4} \alpha_i = 1$, be a functional defined by

$$T(\alpha_1, \alpha_2, \alpha_3; \alpha_4)f(x, y, z) = \sum_{i_1, i_2, i_3} f(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}) ,$$

the sum being over all choices of $(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3})$ from $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. This gives a sum of values of f over a set of points possessing the same symmetries as that of the tetrahedron.

0045-7825/86/$3.50

A tetrahedrally symmetric quadrature formula has the form

$$Qf = \sum_{j=1}^{K} w_j T(\boldsymbol{\alpha}^{(j)})f .$$

We use the word *rule* to refer to a quadrature formula, and call $T(\alpha_1, \alpha_2, \alpha_3; \alpha_4)$ a *basic rule*. Thus a tetrahedrally symmetric rule is a weighted sum of basic rules of type T.

DEFINITION 2.1. If $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ has k distinct nonzero α, with n_i occurrences of the ith α, we say that $T(\alpha_1, \alpha_2, \alpha_3; \alpha_4)$ is of class $[n_1, n_2, \ldots, n_k]$.

Note. In three dimensions, with the constraint $\sum \alpha_i = 1$, there are only 11 different classes of basic rules, viz. [4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1], [3], [2, 1], [1, 1, 1], [2], [1, 1], and [1]. There is only one possible rule of class [4], viz. $T(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}; \frac{1}{4})$, and only one possible rule of each of the classes [3], [2], and [1], namely $T(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; 0)$, $T(\frac{1}{2}, \frac{1}{2}, 0; 0)$, and $T(1, 0, 0; 0)$, respectively. The cost of a rule of class $[n_1, n_2, \ldots, n_k]$ is $4!/n_1!n_2! \cdots n_k!(4 - \sum n_i)!$, and the number of free parameters associated with the basic rule is k, namely $(k-1)$ independent parameters α_i and the weight of the basic rule.

Following [6], we can derive consistency conditions for the formula to be of degree N. These consistency constraints are linear inequalities to be satisfied by the structure of the formula to ensure that the rule can be of degree N. Rather than reproduce the arguments leading to these constraints (which are given in [5, 6]), we give an example with a specific degree. These constraints are necessary and sufficient conditions to ensure that the nonlinear equations for the weights and the α do not satisfy any linear relationship. The hazards of nonlinear relationships and complex solutions remain.

First, we give the conditions a rule must satisfy to have a specified degree. From [2] it is possible to show the following theorem.

THEOREM 2.2. For a rule which is simplicially symmetric to be of degree d over the simplex $0 \leqslant x, y, z, x+y+z \leqslant 1$, it is necessary only that it integrates exactly all polynomials of the form $x^i y^i z^j (1-x-y-z)^k$, for $2i + j + k \leqslant d$, with $i \geqslant j \geqslant k \geqslant 0$.

Now, consider the construction of a rule of degree 5. The polynomials to be integrated are

$$1, \quad xy, \quad x^2y^2, \quad xyz, \quad x^2y^2z, \quad xyz(1-x-y-z).$$

Thus, a formula of degree 5 must integrate six polynomials exactly. Let $K[\boldsymbol{n}]$ be the number of basic rules of class $[\boldsymbol{n}]$ used in the formula. Then there are consistency constraints (derived in [5]) on the numbers $K[\boldsymbol{n}]$ for the formula to have degree 5. In this case, the constraints are

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -2 & -1 & -2 & -3 & -1 & -2 & -2 & -3 & -4 \\ 0 & 0 & 0 & -1 & -2 & -3 & -1 & -2 & -2 & -3 & -4 \\ 0 & 0 & -2 & 0 & 0 & -3 & -1 & -2 & -2 & -3 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & -2 & -3 & -4 \\ 0 & -1 & -2 & 0 & -2 & -3 & 0 & 0 & -2 & -3 & -4 \\ -1 & 0 & -2 & -1 & -2 & -3 & 0 & -2 & -0 & -3 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 & -1 & -2 & -2 & -3 & -4 \\ 0 & 0 & 0 & -1 & -2 & -3 & 0 & -2 & 0 & -3 & -4 \end{bmatrix} \begin{bmatrix} K[1] \\ K[2] \\ K[1,1] \\ K[3] \\ K[2,1] \\ K[1,1,1] \\ K[4] \\ K[3,1] \\ K[2,2] \\ K[2,1,1] \\ K[1,1,1,1] \end{bmatrix} \leqslant \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -6 \\ -3 \\ -1 \\ -1 \\ -1 \\ -3 \\ -1 \\ -1 \end{bmatrix}$$

The cost of a formula with $K[n]$ rules of class $[n]$ is

$$f(\boldsymbol{K}) = 4K[1] + 6K[2] + 12K[1,1] + 4K[3] + 12K[2,1] + 24K[1,1,1] + K[4] \\ + 4K[3,1] + 6K[2,2] + 12K[2,1,1] + 24K[1,1,1,1] .$$

It is then a simple matter of solving the integer programming problem for the vector $\boldsymbol{K}$ which minimizes $f(\boldsymbol{K})$. Successively weaker minima may be found by adding the constraint $f(\boldsymbol{K}) \geqslant$ previous minimum + 1. The minimum point rule of degree 5, which is given, has the form:

$$K[3,1] = 2 , \qquad K[2,2] = 1 .$$

That is, the formula is:

$$w_1 T[\alpha, \alpha, \alpha;\ \beta] + w_2 T[\gamma, \gamma, \gamma;\ \delta] + w_3 T[\lambda, \lambda, \mu;\ \mu] ,$$

where $3\alpha + \beta = 3\gamma + \delta = 2\lambda + 2\mu = 1$. This formula is given in [2] and uses 14 points. The formula given in [4] uses 17 points and comes from the third optimum. The second optimum produces a rule using 15 points, which is listed in Section 3, and appears in [2].

The structures of the minimum point rules for degrees 1 to 10 are given in Appendix A. In the next section we list the rules of degrees 4 to 8.

3. Tetrahedral formulas of degrees 4 to 8

In [2] formulas of degree $2s + 1$ are obtained for the n-simplex, using $(n + s + 1)!/(n + 1)!s!$ points. For $n = 3$ and $s = 1$ this gives a formula of degree 3 using 5 points (the formula given also in [4]) and for $s = 2$ this gives the degree-5 rule in [4]. Appendix A shows structures with fewer points than the rules in [4] for several degrees. In Table 1 we give formulas of degrees 4, 5, 6, 7, and 8. There are two formulas of degree 4, one of which has a negative weight, using 11 and 14 points, respectively. The formula in [4] uses 16 points. The formula of degree 5 appears in [2]. Another formula of degree 5, with the same structure as the one in [2], appears in [8, p. 315].

Table 1
Tetrahedral formulas of degree 4 to 8 ($d+n$ represents an exponent n of 10)

N	M	Weight W	α_1	α_2	α_3	α_4
4	1	−0.131555555555555550$d-1$	0.250000000000000000$d+0$	0.250000000000000000$d+0$	0.250000000000000000$d+0$	0.250000000000000000$d+0$
	4	0.762222222222222222$d-2$	0.714285714285714285$d-1$	0.714285714285714285$d-1$	0.714285714285714285$d-1$	0.785714285714285714$d+0$
	6	0.248888888888888880$d-1$	0.399403576166799219$d+0$	0.399403576166799219$d+0$	0.100596423833200785$d+0$	0.100596423833200785$d+0$
4	6	0.317460317460317450$d-2$	0.500000000000000000$d+0$	0.500000000000000000$d+0$	0. $d+0$	0. $d+0$
	4	0.147649707904967828$d-1$	0.100526765225204467$d+0$	0.100526765225204467$d+0$	0.100526765225204467$d+0$	0.698419704324386603$d+0$
	4	0.221397911142651221$d-1$	0.314372873493192195$d+0$	0.314372873493192195$d+0$	0.314372873493192195$d+0$	0.568813795204234229$d-1$
5	4	0.602678571428571597$d-2$	0.333333333333333333$d+0$	0.333333333333333333$d+0$	0.333333333333333333$d+0$	0. $d+0$
	1	0.302836780970891856$d-1$	0.250000000000000000$d+0$	0.250000000000000000$d+0$	0.250000000000000000$d+0$	0.250000000000000000$d+0$
	4	0.116452490860289742$d-1$	0.909090909090909091$d-1$	0.909090909090909091$d-1$	0.909090909090909091$d-1$	0.727272727272727273$d+0$
	6	0.109491415613864534$d-1$	0.665501535736642813$d-1$	0.665501535736642813$d-1$	0.433449846426335728$d+0$	0.433449846426335728$d+0$
6	4	0.665379170969464506$d-2$	0.214602871259151684$d+0$	0.214602871259151684$d+0$	0.214602871259151684$d+0$	0.356191386222544953$d+0$
	4	0.167953517588677620$d-2$	0.406739585346113397$d-1$	0.406739585346113397$d-1$	0.406739585346113397$d-1$	0.877978124396165982$d+0$
	4	0.922619692394239843$d-2$	0.322337890142275646$d+0$	0.322337890142275646$d+0$	0.322337890142275646$d+0$	0.329863295731730594$d-1$
	12	0.803571428571428248$d-2$	0.636610018750175299$d-1$	0.636610018750175299$d-1$	0.269672331458315867$d+0$	0.603005664791649076$d+0$
7	6	0.970017636684296702$d-3$	0.500000000000000000$d+0$	0.500000000000000000$d+0$	0. $d+0$	0. $d+0$
	1	0.182642234661087939$d-1$	0.250000000000000000$d+0$	0.250000000000000000$d+0$	0.250000000000000000$d+0$	0.250000000000000000$d+0$
	4	0.105999415244141609$d-1$	0.782131923303186549$d-1$	0.782131923303186549$d-1$	0.782131923303186549$d-1$	0.765360423009044044$d+0$
	4	−0.625177401143299494$d-1$	0.121843216663904411$d+0$	0.121843216663904411$d+0$	0.121843216663904411$d+0$	0.634470350008286765$d+0$
	4	0.489142526307353653$d-2$	0.332539164446420554$d+0$	0.332539164446420554$d+0$	0.332539164446420554$d+0$	0.238250666073834549$d-2$
	12	0.275573192239850917$d-1$	0.100000000000000000$d+0$	0.100000000000000000$d+0$	0.200000000000000000$d+0$	0.600000000000000000$d+0$
8	1	0.393270066412926145$d-1$	0.250000000000000000$d+0$	0.250000000000000000$d+0$	0.250000000000000000$d+0$	0.250000000000000000$d+0$
	4	0.408131605934270525$d-2$	0.127470936566639015$d+0$	0.127470936566639015$d+0$	0.127470936566639015$d+0$	0.617587190300082967$d+0$
	4	0.658086773304341943$d-3$	0.320788303926322960$d-1$	0.320788303926322960$d-1$	0.320788303926322960$d-1$	0.903763508822103123$d+0$
	6	0.438425882512284693$d-2$	0.497770956432810185$d-1$	0.497770956432810185$d-1$	0.450222904356718978$d+0$	0.450222904356718978$d+0$
	6	0.138300638425098166$d-1$	0.183730447398549945$d+0$	0.183730447398549945$d+0$	0.316269552601450060$d+0$	0.316269552601450060$d+0$
	12	0.424043742468372453$d-2$	0.231901089397150906$d+0$	0.231901089397150906$d+0$	0.229177878448171174$d-1$	0.513280033360881072$d+0$
	12	0.223873973961420164$d-2$	0.379700484718286102$d-1$	0.379700484718286102$d-1$	0.730313427807538396$d+0$	0.193746475248804382$d+0$

As far as we know, the formulas of degrees 4, 6, 7, and 8 are new. The presence of negative weights in the formulas of degrees 4, 7, and 8 causes no serious problem, since in both cases the sum of the absolute values of the weights when applied to $f \equiv 1$ is less than 5, indicating that the round-off error accumulation is minimal. (see [1, p. 208], where it is shown that the relative round-off error is bounded by the sum of the absolute values of the weights.)

Appendix A

Feasible structures for simplicially symmetric formulas over the tetrahedron are listed. A set of 15 integers is given for each structure, namely,

$$N, i, j, K[1], K[2], K[1,1], K[3], K[2,1], K[1,1,1], K[4], K[3,1], K[2,2], K[2,1,], K[1,1,1,1], M,$$

where:

N is the degree of the formula;
i represents the ith consecutive optimum structure;
j represents the jth equal cost structure at the ith optimum;
K are the rule structure parameters;
M is the cost of the formula.

N	i	j						K						M
1	1	1	0	0	0	0	0	0	1	0	0	0	0	1
1	2	1	1	0	0	0	0	0	0	0	0	0	0	4
1	2	2	0	0	0	1	0	0	0	0	0	0	0	4
1	2	3	0	0	0	0	0	0	0	1	0	0	0	4
1	3	1	1	0	0	0	0	0	1	0	0	0	0	5
1	3	2	0	0	0	1	0	0	1	0	0	0	0	5
1	3	3	0	0	0	0	0	0	1	1	0	0	0	5
1	4	1	0	1	0	0	0	0	0	0	0	0	0	6
1	4	2	0	0	0	0	0	0	0	0	1	0	0	6
1	5	1	0	1	0	0	0	0	1	0	0	0	0	7
1	5	2	0	0	0	0	0	0	1	0	1	0	0	7
2	1	1	0	0	0	0	0	0	0	1	0	0	0	4
2	2	1	1	0	0	0	0	0	1	0	0	0	0	5
2	2	2	0	0	0	1	0	0	1	0	0	0	0	5
2	2	3	0	0	0	0	0	0	1	1	0	0	0	5
2	3	1	0	0	0	0	0	0	0	0	1	0	0	6
2	4	1	0	1	0	0	0	0	1	0	0	0	0	7
2	4	2	0	0	0	0	0	0	1	0	1	0	0	7
2	5	1	1	0	0	1	0	0	0	0	0	0	0	8
2	5	2	1	0	0	0	0	0	0	1	0	0	0	8
2	5	3	0	0	0	1	0	0	0	1	0	0	0	8
2	5	4	0	0	0	0	0	0	0	2	0	0	0	8
3	1	1	0	0	0	0	0	0	1	1	0	0	0	5
3	2	1	1	0	0	0	0	0	0	1	0	0	0	8
3	2	2	0	0	0	1	0	0	0	1	0	0	0	8
3	2	3	0	0	0	0	0	0	0	2	0	0	0	8

N	i	j						K						M
3	3	1	1	0	0	1	0	0	1	0	0	0	0	9
3	3	2	1	0	0	0	0	0	1	1	0	0	0	9
3	3	3	0	0	0	1	0	0	1	1	0	0	0	9
3	3	4	0	0	0	0	0	0	1	2	0	0	0	9
3	4	1	0	1	0	0	0	0	0	1	0	0	0	10
3	4	2	1	0	0	0	0	0	0	0	1	0	0	10
3	4	3	0	0	0	1	0	0	0	0	1	0	0	10
3	4	4	0	0	0	0	0	0	0	1	1	0	0	10
3	5	1	1	1	0	0	0	0	1	0	0	0	0	11
3	5	2	0	1	0	1	0	0	1	0	0	0	0	11
3	5	3	0	1	0	0	0	0	1	1	0	0	0	11
3	5	4	1	0	0	0	0	0	1	0	1	0	0	11
3	5	5	0	0	0	1	0	0	1	0	1	0	0	11
3	5	6	0	0	0	0	0	0	1	1	1	0	0	11
4	1	1	0	0	0	0	0	0	1	1	1	0	0	11
4	2	1	0	1	0	0	0	0	0	2	0	0	0	14
4	2	2	1	0	0	0	0	0	0	1	1	0	0	14
4	2	3	0	0	0	1	0	0	0	1	1	0	0	14
4	2	4	0	0	0	0	0	0	0	2	1	0	0	14
4	3	1	1	1	0	0	0	0	1	1	0	0	0	15
4	3	2	0	1	0	1	0	0	1	1	0	0	0	15
4	3	3	0	1	0	0	0	0	1	2	0	0	0	15
4	3	4	1	0	0	1	0	0	1	0	1	0	0	15
4	3	5	1	0	0	0	0	0	1	1	1	0	0	15
4	3	6	0	0	0	1	0	0	1	1	1	0	0	15
4	3	7	0	0	0	0	0	0	1	2	1	0	0	15
4	4	1	0	1	0	0	0	0	0	1	1	0	0	16
4	4	2	0	0	0	0	0	0	0	1	2	0	0	16
4	4	3	0	0	0	0	0	0	0	1	0	1	0	16
4	5	1	0	0	1	0	0	0	1	1	0	0	0	17
4	5	2	0	0	0	0	1	0	1	1	0	0	0	17
4	5	3	0	1	0	0	0	0	1	1	1	0	0	17
4	5	4	0	0	0	0	0	0	1	1	2	0	0	17
4	5	5	1	0	0	0	0	0	1	0	0	1	0	17
4	5	6	0	0	0	1	0	0	1	0	0	1	0	17
4	5	7	0	0	0	0	0	0	1	1	0	1	0	17
5	1	1	0	0	0	0	0	0	0	2	1	0	0	14
5	2	1	0	1	0	0	0	0	1	2	0	0	0	15
5	2	2	1	0	0	0	0	0	1	1	1	0	0	15
5	2	3	0	0	0	1	0	0	1	1	1	0	0	15
5	2	4	0	0	0	0	0	0	1	2	1	0	0	15
5	3	1	0	0	0	0	0	0	1	1	0	1	0	17
5	4	1	1	1	0	0	0	0	0	2	0	0	0	18
5	4	2	0	1	0	1	0	0	0	2	0	0	0	18
5	4	3	0	1	0	0	0	0	0	3	0	0	0	18
5	4	4	1	0	0	1	0	0	0	1	1	0	0	18
5	4	5	1	0	0	0	0	0	0	2	1	0	0	18
5	4	6	0	0	0	1	0	0	0	2	1	0	0	18
5	4	7	0	0	0	0	0	0	0	3	1	0	0	18
5	5	1	1	1	0	1	0	0	1	1	0	0	0	19
5	5	2	1	1	0	0	0	1	1	2	0	0	0	19

N	i	j	K											M
5	5	3	0	1	0	1	0	0	1	2	0	0	0	19
5	5	4	0	1	0	0	0	0	1	3	0	0	0	19
5	5	5	1	0	0	1	0	0	1	1	1	0	0	19
5	5	6	1	0	0	0	0	0	1	2	1	0	0	19
5	5	7	0	0	0	1	0	0	1	2	1	0	0	19
5	5	8	0	0	0	0	0	0	1	3	1	0	0	19
5	5	9	0	0	0	0	0	0	1	0	1	1	0	19
6	1	1	0	1	0	0	0	0	0	3	1	0	0	24
6	1	2	1	0	0	0	0	0	0	2	2	0	0	24
6	1	3	0	0	0	1	0	0	0	2	2	0	0	24
6	1	4	0	0	0	0	0	0	0	3	2	0	0	24
6	1	5	0	0	0	0	0	0	0	3	0	1	0	24
6	2	1	1	1	0	0	0	0	1	2	1	0	0	25
6	2	2	0	1	0	1	0	0	1	2	1	0	0	25
6	2	3	0	1	0	0	0	0	1	3	1	0	0	25
6	2	4	1	0	0	0	0	0	1	2	2	0	0	25
6	2	5	0	0	0	1	0	0	1	2	2	0	0	25
6	2	6	0	0	0	0	0	0	1	3	2	0	0	25
6	2	7	1	0	0	0	0	0	1	2	0	1	0	25
6	2	8	0	0	0	1	0	0	1	2	0	1	0	25
6	2	9	0	0	0	0	0	0	1	3	0	1	0	25
6	3	1	0	0	0	0	0	0	0	2	1	1	0	26
6	4	1	0	0	1	0	0	0	1	2	1	0	0	27
6	4	2	0	0	0	0	1	0	1	2	1	0	0	27
6	4	3	0	1	0	0	0	0	1	2	0	1	0	27
6	4	4	1	0	0	0	0	0	1	1	1	1	0	27
6	4	5	0	0	0	1	0	0	1	1	1	1	0	27
6	4	6	0	0	0	0	0	0	1	2	1	1	0	27
6	5	1	1	1	0	1	0	0	0	2	1	0	0	28
6	5	2	1	1	0	0	0	0	0	3	1	0	0	28
6	5	3	0	1	0	1	0	0	0	3	1	0	0	28
6	5	4	0	1	0	0	0	0	0	4	1	0	0	28
6	5	5	1	0	0	1	0	0	0	2	2	0	0	28
6	5	6	1	0	0	0	0	0	0	3	2	0	0	28
6	5	7	0	0	0	1	0	0	0	3	2	0	0	28
6	5	8	0	0	0	0	0	0	0	4	2	0	0	28
6	5	9	1	0	0	1	0	0	0	2	0	1	0	28
6	5	10	1	0	0	0	0	0	0	3	0	1	0	28
6	5	11	0	0	0	1	0	0	0	3	0	1	0	28
6	5	12	0	0	0	0	0	0	0	4	0	1	0	28
6	5	13	0	0	0	0	0	0	0	1	2	1	0	28
7	1	1	1	0	0	0	0	0	0	3	2	0	0	28
7	1	2	0	0	0	1	0	0	0	3	2	0	0	28
7	1	3	0	0	0	0	0	0	0	4	2	0	0	28
7	2	1	1	0	0	0	0	0	1	3	2	0	0	29
7	2	2	0	0	0	1	0	0	1	3	2	0	0	29
7	2	3	0	0	0	0	0	0	1	4	2	0	0	29
7	3	1	0	0	0	0	0	0	0	3	1	1	0	30
7	4	1	0	0	1	0	0	0	1	3	1	0	0	31
7	4	2	0	0	0	0	1	0	1	3	1	0	0	31
7	4	3	0	1	0	0	0	0	1	3	0	1	0	31

N	i	j						K					M
7	4	4	1	0	0	0	0	0	1	2	1	1	0	31
7	4	5	0	0	0	1	0	0	1	2	1	1	0	31
7	4	6	0	0	0	0	0	0	1	3	1	1	0	31
7	5	1	1	0	0	1	0	0	0	3	2	0	0	32
7	5	2	1	0	0	0	0	0	0	4	2	0	0	32
7	5	3	0	0	0	1	0	0	0	4	2	0	0	32
7	5	4	0	0	0	0	0	0	0	5	2	0	0	32
7	5	5	0	0	0	0	0	0	0	2	2	1	0	32
8	1	1	0	0	0	0	0	0	0	4	2	1	0	40
8	2	1	1	0	0	0	0	0	1	3	2	1	0	41
8	2	2	0	0	0	1	0	0	1	3	2	1	0	41
8	2	3	0	0	0	0	0	0	1	4	2	1	0	41
8	3	1	0	0	0	0	0	0	1	3	1	2	0	43
8	4	1	1	0	0	1	0	0	0	3	2	1	0	44
8	4	2	1	0	0	0	0	0	0	4	2	1	0	44
8	4	3	0	0	0	1	0	0	0	4	2	1	0	44
8	4	4	0	0	0	0	0	0	0	5	2	1	0	44
8	5	1	1	0	0	1	0	0	1	3	2	1	0	45
8	5	2	1	0	0	0	0	0	1	4	2	1	0	45
8	5	3	0	0	0	1	0	0	1	4	2	1	0	45
8	5	4	0	0	0	0	0	0	1	5	2	1	0	45
8	5	5	0	0	0	0	0	0	1	2	2	2	0	45
9	1	1	0	0	0	0	0	0	0	4	2	2	0	52
9	2	1	0	0	1	0	0	0	1	4	2	1	0	53
9	2	2	0	0	0	0	1	0	1	4	2	1	0	53
9	2	3	0	1	0	0	0	0	1	4	1	2	0	53
9	2	4	1	0	0	0	0	0	1	3	2	2	0	53
9	2	5	0	0	0	1	0	0	1	3	2	2	0	53
9	2	6	0	0	0	0	0	0	1	4	2	2	0	53
9	2	7	0	0	0	0	0	0	1	4	0	3	0	53
9	3	1	0	0	0	0	0	0	1	3	1	3	0	55
9	4	1	1	0	1	0	0	0	0	4	2	1	0	56
9	4	2	0	0	1	1	0	0	0	4	2	1	0	56
9	4	3	1	0	0	0	1	0	0	4	2	1	0	56
9	4	4	0	0	0	1	1	0	0	4	2	1	0	56
9	4	5	0	0	1	0	0	0	0	5	2	1	0	56
9	4	6	0	0	0	0	1	0	0	5	2	1	0	56
9	4	7	1	1	0	0	0	0	0	4	1	2	0	56
9	4	8	0	1	0	1	0	0	0	4	1	2	0	56
9	4	9	0	1	0	0	0	0	0	5	1	2	0	56
9	4	10	1	0	0	1	0	0	0	3	2	2	0	56
9	4	11	1	0	0	0	0	0	0	4	2	2	0	56
9	4	12	0	0	0	1	0	0	0	4	2	2	0	56
9	4	13	0	0	0	0	0	0	0	5	2	2	0	56
9	4	14	1	0	0	0	0	0	0	4	0	3	0	56
9	4	15	0	0	0	1	0	0	0	4	0	3	0	56
9	4	16	0	0	0	0	0	0	0	5	0	3	0	56
9	5	1	1	0	1	1	0	0	1	3	2	1	0	57
9	5	2	1	0	0	1	1	0	1	3	2	1	0	57
9	5	3	1	0	1	0	0	0	1	4	2	1	0	57
9	5	4	0	0	1	1	0	0	1	4	2	1	0	57

N	i	j						K						M
9	5	5	1	0	0	0	1	0	1	4	2	1	0	57
9	5	6	0	0	0	1	1	0	1	4	2	1	0	57
9	5	7	0	0	1	0	0	0	1	5	2	1	0	57
9	5	8	0	0	0	0	1	0	1	5	2	1	0	57
9	5	9	1	1	0	1	0	0	1	3	1	2	0	57
9	5	10	1	1	0	0	0	0	1	4	1	2	0	57
9	5	11	0	1	0	1	0	0	1	4	1	2	0	57
9	5	12	0	1	0	0	0	0	1	5	1	2	0	57
9	5	13	1	0	0	1	0	0	1	3	2	2	0	57
9	5	14	1	0	0	0	0	0	1	4	2	2	0	57
9	5	15	0	0	0	1	0	0	1	4	2	2	0	57
9	5	16	0	0	0	0	0	0	1	5	2	2	0	57
9	7	17	1	0	0	1	0	0	1	3	0	3	0	57
9	5	18	1	0	0	0	0	0	1	4	0	3	0	57
9	5	19	0	0	0	1	0	0	1	4	0	3	0	57
9	5	20	0	0	0	0	0	0	1	5	0	3	0	57
9	5	21	0	0	0	0	0	0	1	2	2	3	0	57
10	1	1	0	0	0	0	0	0	0	5	2	3	0	68
10	2	1	1	0	0	0	0	0	1	4	2	3	0	69
10	2	2	0	0	0	1	0	0	1	4	2	3	0	69
10	2	3	0	0	0	0	0	0	1	5	2	3	0	69
10	3	1	0	0	0	0	0	0	0	4	3	3	0	70
10	4	1	0	1	0	0	0	0	1	4	2	3	0	71
10	4	2	0	0	0	0	0	0	1	4	3	3	0	71
10	4	3	0	0	0	0	0	0	1	4	1	4	0	71
10	5	1	1	0	0	1	0	0	0	4	2	3	0	72
10	5	2	1	0	0	0	0	0	0	5	2	3	0	72
10	5	3	0	0	0	1	0	0	0	5	2	3	0	72
10	5	4	0	0	0	0	0	0	0	6	2	3	0	72

Acknowledgment

This research was supported by grant A2699 from the Natural Sciences and Engineering Research Council of Canada. The calculations were carried out on the VAX 780 of the Department of Mathematics, Statistics and Computing Science, Dalhousie University. The author wishes to thank the referees for their comments and criticisms, and Dr. Graeme Fairweather for helpful discussions.

References

[1] P.J. Davis and P. Rabinowitz, Methods of Numerical Integration (Academic Press, New York, 1975) 208–213.

[2] A. Grundman and M.M. Möller, Invariant integration formulas for the n-simplex by combinatorial methods, SIAM J. Numer. Anal. 15(2) (1978) 282–290.

[3] P.C. Hammer, O.J. Marlowe and A.H. Stroud, Numerical integration over simplexes and cones, Math. Tables Aids Comput. 10 (1956) 130–137.

[4] Y. Jinyun, Symmetric Gaussian quadrature formulae for tetrahedronal regions, Comput. Meths. Appl. Mech. Engrg. 43 (1984) 349–353.
[5] P. Keast, Cubature formulas for the surface of the sphere, J. Comput. Appl. Math. (1986) Special Issue on Quadrature.
[6] P. Keast and J.C. Diaz, Fully symmetric integration formulas for the surface of the sphere in s dimensions, SIAM J. Numer. Anal. 20(2) (1983) 406–419.
[7] P. Silvester, Newton–Cotes quadrature formulae for N-dimensional simplexes, in: Proceedings Second Canadian Congress on Applied Mechanics, Waterloo, Canada, 1969.
[8] A.H. Stroud, Approximate Calculation of Multiple Integrals (Prentice-Hall, Englewood Cliffs, NJ, 1971) 306–315.
[9] O.C. Zienkiewicz, The Finite Element Method in Engineering Science (McGraw-Hill, London, 3rd ed., 1977).